

On time transformations for differential equations with state-dependent delay

A.V. REZOUNENKO

Kharkiv National University, Department of Mechanics and Mathematics,
4, Svobody Sqr., Kharkiv, 61022, Ukraine
E-mail: rezounenko@yahoo.com

Systems of differential equations with state-dependent delay are considered. The delay dynamically depends on the state i.e. is governed by an additional differential equation. By applying the time transformations we arrive to constant delay systems and compare the asymptotic properties of the original and transformed systems.

1. Introduction

Delay differential equations successfully used to model and study a number of applied problems in physics, biology, chemistry. Basic mathematical theory of constant delay equations can be found in the classical monographs [5, 6] and references therein.

Differential equations with state-dependent delay (SDD) attracted much attention during last decades and many deep results have been obtained so far (see [9, 11, 15, 10] and references therein). Such equations with discrete state-dependent delays are always *nonlinear* by their nature. As described in [9], this type of delay brings additional difficulties in proving such basic properties of solutions as uniqueness and continuous dependence on initial data. The main approach to get the well-posed initial-value problem is to restrict the set of initial functions and hence the set of solutions to C^1 -functions [9]. For an alternative approach where an additional condition on the SDD is used to get a well-posed initial-value problem in the space of continuous functions see [12, 13]. In this note we rely on the classical approach [9] and compare the SDD problem with another one with a constant delay. This constant delay problem is constructed by using the so-called *time transformations* [2, 3]. This transformation could be applied to any particular solution along which the deviating argument is monotone. To be assured that the monotonicity holds for all solutions we concentrate on the system when the state-dependent delay is governed by an additional differential equation and provide a sufficient condition for the monotonicity of the deviating argument. This type of equations is used to describe some models of population dynamics, see [1] and references therein. In [1] one could also find motivations to study this type of SDD and comparison with the frequently used case when SDD is presented by explicit or implicit functionals.

Our main goal in this note is to compare the asymptotic properties of the SDD system with the corresponding ones of the system after the time transformation. We try to find conditions which guarantee that such properties as stability, boundedness and compactness of the initial SDD problem survives under the time transformations.

2. Time transformations

We study the following non-autonomous system with state-dependent delay (see the autonomous case in [1])

$$\dot{y}(t) = f(t, y(t), y(t - \eta(t))), \quad t > t^0, \quad (1)$$

$$\dot{\eta}(t) = -\mu(\eta(t) - \tilde{\eta}) + G(y(t)), \quad t > t^0, \quad (2)$$

with initial data

$$y(t) = g(t), \quad t \in [t^0 - h, t^0], \quad (3)$$

$$\eta(t^0) = \eta^0. \quad (4)$$

Here $y \in R^m, \eta \in R, \eta^0, \tilde{\eta} > 0$. The function η is a state-dependent delay since it is a solution of the equation (2) where there is a dependence on y .

In the sequel we will denote $h \equiv 2\tilde{\eta}$ and also $X \equiv C^1([-h, 0]; R^m) \times R$ with the natural norm.

Lemma 1. *Let f and G be Lipschitz and $|G(y)| \leq \mu\tilde{\eta}$ for all $y \in R^m$. Then for any $g \in C^1([t^0 - h, t^0]; R^m)$, $\eta(t^0) \equiv \eta^0 \in [0, 2\tilde{\eta}]$ the system (1)-(4) has a unique global solution such that $\eta(t) \in [0, 2\tilde{\eta}]$ for all $t \geq t^0$. The solution continuously depends on initial data $(g; \eta^0)$.*

Proof of lemma 1. The existence of solutions is simple since the righthand sides of equations (1), (2) are continuous. Solutions are global due to Lipschitz properties of f and G . The uniqueness follows from the well-known results on the state-dependent delay equations (see e.g. [9]) since we consider Lipschitz initial function g .

Using the property $|G(y)| \leq \mu\tilde{\eta}$, one can easily show that (for any y) any solution of (2), (4) satisfies $\eta(t) \in [0, 2\tilde{\eta}]$ provided $\eta(t^0) \equiv \eta^0 \in [0, 2\tilde{\eta}]$.

Now we show the continuous dependence on initial data. For the simplicity of presentation we put $t^0 = 0$. Let us consider a pair $(\bar{g}; \bar{\eta}^0) \in C^1([-h, 0]; R^m) \times [0, h] \subset X$ and an arbitrary sequence $(g^n; \eta^{0,n})$ such that $\|(g^n; \eta^{0,n}) - (\bar{g}; \bar{\eta}^0)\|_X \rightarrow 0$ as $n \rightarrow \infty$.

We rewrite the system (1)-(4) in the integral form

$$y^n(t) = g^n(0) + \int_0^t f(s, y^n(s), y^n(s - \eta^n(s))) ds, \quad (5)$$

$$\eta^n(t) - \tilde{\eta} = e^{-\mu t}(\eta^{0,n} - \tilde{\eta}) + \int_0^t e^{-\mu(t-s)} G(y^n(s)) ds. \quad (6)$$

Similar equations are for initial data $(\bar{g}; \bar{\eta}^0)$. For the differences of solutions, using the Lipschitz properties of f and G (the corresponding Lipschitz constants are denoted by L_f and L_G) we have

$$\begin{aligned} |y^n(t) - \bar{y}(t)| &\leq |g^n(0) - \bar{g}(0)| \\ &+ L_f \int_0^t \{|y^n(s - \eta^n(s)) - \bar{y}(s - \eta^n(s))| + |\bar{y}(s - \eta^n(s)) - \bar{y}(s - \bar{\eta}(s))| + |y^n(s) - \bar{y}(s)|\} ds, \\ |\eta^n(t) - \bar{\eta}(t)| &\leq e^{-\mu t}|\eta^{0,n} - \bar{\eta}^0| + L_G \int_0^t |y^n(s) - \bar{y}(s)| ds. \end{aligned}$$

Let us fix any $T > 0$. Since all solutions are C^1 in time (see e.g. [9]) for $t > 0$ we can denote by $L_{\bar{y}, T}$ the Lipschitz constant of the solution $\bar{y}(t)$, $t \in [-h, T]$. Hence $|\bar{y}(s - \eta^n(s)) - \bar{y}(s - \bar{\eta}(s))| \leq L_{\bar{y}, T}|\eta^n(s) - \bar{\eta}(s)|$ for all $s \in [0, t] \subset [0, T]$. Denoting for short $\beta^n(t) \equiv \max_{\tau \in [0, t]} \{|y^n(\tau) - \bar{y}(\tau)| + |\eta^n(\tau) - \bar{\eta}(\tau)|\}$ and $C^T \equiv \max\{2L_f + L_G; L_{\bar{y}, T}\}$ we obtain

$$0 \leq \beta^n(t) \leq \beta^n(0) + L_f \max_{\tau \in [-h, 0]} |g^n(\tau) - \bar{g}(\tau)| + C^T \int_0^t \beta^n(s) ds.$$

We apply the Gronwall's inequality and get for all $t \in [0, T]$

$$\begin{aligned} &\max_{\tau \in [0, t]} \{|y^n(\tau) - \bar{y}(\tau)| + |\eta^n(\tau) - \bar{\eta}(\tau)|\} \\ &\leq \left\{ (1 + L_f) \max_{\tau \in [-h, 0]} |g^n(\tau) - \bar{g}(\tau)| + |\eta^{0,n} - \bar{\eta}^0| \right\} \exp(t \cdot \max\{2L_f + L_G; L_{\bar{y}, T}\}). \end{aligned}$$

The last convergence and equations (1), (2) give a similar estimate (convergence) for the time derivatives, so we finally get

$$\|y^n - \bar{y}\|_{C^1[0, t]} + \|\eta^n - \bar{\eta}\|_{C^1[0, t]} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

This gives the continuous dependence on initial data and completes the proof of lemma 1. ■

For any solution $(y; \eta)$ of the system (1)-(4) we call the function σ given by

$$\sigma(t) = t - \eta(t), \quad t \geq t^0 \quad (7)$$

the *deviating argument* of $(y; \eta)$.

Our goal is to investigate properties connected to the time transformation approach introduced in [2, 3]. We are going to use a function $t = \alpha(s)$ called *time transformation* [3] to convert a particular solution $(y; \eta)$ of the system (1)-(4) into a solution $(z; \chi; \alpha)$ of *constant delay system*

$$\begin{cases} \dot{z}(s) &= f(\alpha(s), z(s), z(s - h)) \dot{\alpha}(s), \quad s \geq s^0, \\ z(s) &= g(\omega(s)), \quad s \in [s^0 - h, s^0], \\ \dot{\chi}(s) &= -\mu(\chi(s) - \tilde{\eta}) \dot{\alpha}(s) + G(z(s)) \dot{\alpha}(s), \\ \chi(s^0) &= \eta^0, \end{cases} \quad (8)$$

where α satisfies the algebraic equation

$$\begin{cases} \alpha(s) - \chi(s) &= \alpha(s - h), \quad s \geq s^0, \\ \alpha(s) &= \omega(s), \quad s \in [s^0 - h, s^0], \end{cases} \quad (9)$$

Here $\omega : [s^0 - h, s^0] \rightarrow R$ is an arbitrary C^1 -function with positive derivative and such that $\omega(s^0 - h) = \omega(s^0) - \eta^0 < t^0, \omega(s^0) = t^0$.

Remark. We notice that equation (9) is different from the corresponding rules used in [3] and [2] since here we have no a given **lag** function.

The time transformation α is constructed step by step (see (9), (7)) by the rule

$$\alpha(s) = \sigma^{-1}(\alpha(s - h)), \quad s \in [s^0 + (k - 1)h, s^0 + kh], \quad k = 0, 1, 2, \dots \quad (10)$$

Here we used (see (7)) $\sigma(\alpha(s)) = \alpha(s) - \eta(\alpha(s)) = \alpha(s) - \chi(s) = \alpha(s - h)$, since $\chi(s) = \eta(\alpha(s))$.

It is clear that one needs the invertibility of σ to define α . In [3] the rule (10) was used *assuming* that $\dot{\sigma}(t) > 0$ (or $\dot{\sigma}(t) < 0$). More precisely, it was used for those solutions along which $\dot{\sigma}(t) > 0$ (or $\dot{\sigma}(t) < 0$). In our study we can give a simple condition which *guarantees* that along *all* solutions we have $\dot{\sigma}(t) > 0$ and hence σ is always invertible. Such a simple condition is $2\mu\tilde{\eta} < 1$. It is easy to see using (2) that in this case $|\dot{\eta}(t)| \leq \mu|\eta - \tilde{\eta}| + |G(y(t))| \leq \mu\tilde{\eta} + \mu\tilde{\eta} < 1$. Here we used the assumption $|G(y)| \leq \mu\tilde{\eta}$. Now (7) implies $\dot{\sigma}(t) = 1 - \dot{\eta}(t) > 0$.

Remark. Taking into account that the state-dependent delay η takes values in $[0, 2\tilde{\eta}] = [0, h]$, one could say that the assumption $2\mu\tilde{\eta} < 1$ means the "slow changing of delay" in the range $[0, 2\tilde{\eta}]$.

It is important that if $\dot{\sigma}(t) > 0$, then $\dot{\alpha}(s) > 0$ since on the initial time segment $\dot{\alpha}(s) = \dot{\omega}(s) > 0, s \in [s^0 - h, s^0]$ and $\dot{\alpha}(s) = \frac{d\sigma^{-1}(\alpha(s-h))}{ds} = \frac{1}{\sigma'(\alpha(s-h))} \dot{\alpha}(s-h) = \frac{\dot{\alpha}(s-h)}{\sigma'(\sigma(\alpha(s)))} > 0$ step by step (see also [3, p.28]). Here we used $\sigma(\alpha(s)) = \alpha(s - h)$ (see (9), (10)).

By construction (see [3, propositions 1,2]) the connection between a solution (y, η) of (1)-(4) and the corresponding solution (z, χ, α) of (8)-(9) is given by

$$y(t) = z(\alpha^{-1}(t)), \quad t \in [t^0 - h, +\infty), \quad z(s) = y(\alpha(s)), \quad \chi(s) = \eta(\alpha(s)), \quad s \in [s^0 - h, +\infty), \quad t = \alpha(s), \quad t^0 = \alpha(s^0). \quad (11)$$

Lemma 2. Let f and G be as in lemma 1. Consider a sequence $\{(g^n; \eta^{0,n})\}$ such that $\|(g^n; \eta^{0,n}) - (\bar{g}; \bar{\eta}^0)\|_{C^1([t^0-h, t^0]; R^m) \times R} \rightarrow 0$ and a sequence $\{\omega^n\}$ such that $\|\omega^n - \bar{\omega}\|_{C^1[t^0-h, t^0]} \rightarrow 0$ as $n \rightarrow \infty$. Then for any $S > 0$ the sequence of time transformations α^n uniformly converges to $\bar{\alpha}$ i.e. $\max_{s \in [s^0, s^0+S]} |\alpha^n(s) - \bar{\alpha}(s)| \rightarrow 0$ as $n \rightarrow \infty$.

Proof of lemma 2. First, using (7), we have from lemma 1 that $\max_{t \in [t^0, t^0+T]} |\sigma^n(t) - \bar{\sigma}(t)| \rightarrow 0$ as $n \rightarrow \infty$. This convergence and (2) imply that $\|\sigma^n - \bar{\sigma}\|_{C^1[t^0, t^0+T]} \rightarrow 0$.

Since $\frac{d}{dt} \bar{\sigma}(t) > 0$ on $[t^0, t^0 + T]$, then there is a $\bar{\delta} > 0$ such that $\frac{d}{dt} \bar{\sigma}(t) \geq 2\bar{\delta} > 0$ for all $t \in [t^0, t^0 + T]$. This and $\|\sigma^n - \bar{\sigma}\|_{C^1[t^0, t^0+T]} \rightarrow 0$ give

$$\frac{d}{dt} \sigma^n(t) \geq \bar{\delta} > 0 \text{ for all } t \in [t^0, t^0 + T] \text{ and } n > n_1. \quad (12)$$

Using the definition of α (see (10)) and the convergence $\omega^n \rightarrow \bar{\omega}$ as $n \rightarrow \infty$ we only need to show the property $\max_{\tau \in [0, T]} |(\sigma^n)^{-1}(\tau) - (\bar{\sigma})^{-1}(\tau)| \rightarrow 0$ as $n \rightarrow \infty$. Let us denote by $\gamma^n(s) = (\sigma^n)^{-1}(s), \bar{\gamma}(s) = (\bar{\sigma})^{-1}(s)$. Assume opposite i.e. $\gamma^n(s) \not\rightarrow \bar{\gamma}(s)$ on some $[s^0, s^0 + S]$. Hence

$$\exists \varepsilon_0 > 0 : \forall N \in \mathbf{N} \quad \exists n_N \geq N \quad \exists s_{n_N} \in [s^0, s^0 + S] : |\gamma^n(s_{n_N}) - \bar{\gamma}(s_{n_N})| \geq \varepsilon_0.$$

Considering $N = 1, 2, \dots$ we get two sequences $\{n_k\}_{k=1}^\infty$ and $\{s_{n_k}\}_{k=1}^\infty \subset [s^0, s^0 + S]$ such that

$$|\gamma^{n_k}(s_{n_k}) - \bar{\gamma}(s_{n_k})| \geq \varepsilon_0. \quad (13)$$

Since $[s^0, s^0 + S]$ is compact we have $\hat{s} \in [s^0, s^0 + S]$ and a subsequence again denoted by $\{n_k\}_{k=1}^\infty$ such that $s_{n_k} \rightarrow \hat{s} \in [s^0, s^0 + S]$. We can write

$$\gamma^{n_k}(s_{n_k}) - \bar{\gamma}(s_{n_k}) = (\gamma^{n_k}(s_{n_k}) - \gamma^{n_k}(\hat{s})) + (\gamma^{n_k}(\hat{s}) - \bar{\gamma}(\hat{s})) + (\bar{\gamma}(\hat{s}) - \bar{\gamma}(s_{n_k}))$$

The last term vanishes due the continuity of $\bar{\gamma}$, the second one due to the point-wise convergence $\gamma^n(s) \rightarrow \bar{\gamma}(s)$ for all $s \in [s^0, s^0 + S]$. Hence the only possibility to satisfy (13) is that there is an integer k_1 such that for all $k \geq k_1$ one has

$$|\gamma^{n_k}(s_{n_k}) - \gamma^{n_k}(\hat{s})| > \varepsilon_0/2.$$

The last property together with $s_{n_k} \rightarrow \hat{s}$ and differentiability of all γ^{n_k} imply that the derivatives $\frac{d}{ds}\gamma^{n_k}$ are unbounded in a neighborhood of \hat{s} . This contradicts the property $\frac{d}{dt}\sigma^n(t) \geq \bar{\delta} > 0$ (see (12)) since $\frac{d}{ds}\gamma^n(s) = \frac{1}{\frac{d}{dt}\sigma^n(t)}$. The proof of lemma 2 is complete. ■

Now combining lemmata 1 and 2 (and property $\mu\tilde{\eta} < \frac{1}{2}$) we can formulate the first result on the continuous dependence of the time transformation on initial data.

Theorem 1. *Let f and G be Lipschitz and $|G(y)| \leq \mu\tilde{\eta} < \frac{1}{2}$ for all $y \in R^m$. Consider a sequence $\{(g^n; \eta^{0,n})\}$ such that $\|(g^n; \eta^{0,n}) - (\bar{g}; \bar{\eta}^0)\|_{C^1([t^0-h, t^0]; R^m) \times R} \rightarrow 0$ and a sequence $\{\omega^n\}$ such that $\|\omega^n - \bar{\omega}\|_{C^1[t^0-h, t^0]} \rightarrow 0$ as $n \rightarrow \infty$. Then the time transformation gives the sequence of the corresponding solutions $\{(z^n, \chi^n, \alpha^n)\}$ of the constant delay system (8)-(9) (see (11)) such that for any $S > 0$ one has*

$$\max_{s \in [s^0, s^0 + S]} \{|z^n(s) - \bar{z}(s)| + |\chi^n(s) - \bar{\chi}(s)| + |\alpha^n(s) - \bar{\alpha}(s)|\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

and $\dot{\alpha}^n(s) > 0$ for $s \in [s^0, s^0 + S]$.

Remark. We should notice that (8), (9) is the system of coupled differential and algebraic equations. It is necessary to comment on how to solve it. The way is different from the one of [3, 2] since we have no a given lag function (c.f. [3, Section 2.1]). Using (9), we write for $s \in [0, h]$ (and than continue step by step with the step h): $\alpha(s) = \chi(s) + \omega(s-h)$. Then we substitute it into the differential equation for χ in (8) to get $\dot{\chi}(s) = -\mu(\chi(s) - \tilde{\eta})(\dot{\chi}(s) + \dot{\omega}(s-h)) + G(z(s))(\dot{\chi}(s) + \dot{\omega}(s-h))$. Hence $\dot{\chi}(s)[1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))] = \{-\mu(\chi(s) - \tilde{\eta}) + G(z(s))\} \dot{\omega}(s-h)$. We remind that the assumption $2\mu\tilde{\eta} < 1$ implies $|\mu(\chi(s) - \tilde{\eta}) - G(z(s))| < 1$. It gives

$$\dot{\chi}(s) = \{-\mu(\chi(s) - \tilde{\eta}) + G(z(s))\} \dot{\omega}(s-h) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1}. \quad (14)$$

Now to rewrite the first equation in (8) we use again $\alpha(s) = \chi(s) + \omega(s-h)$ (and $\dot{\alpha}(s) = \dot{\chi}(s) + \dot{\omega}(s-h)$), substitute it in (8) and use (14). It gives $\dot{z}(s) = f(\alpha(s), z(s), z(s-h)) \dot{\alpha}(s) = f(\chi(s) + \omega(s-h), z(s), z(s-h))(\dot{\chi}(s) + \dot{\omega}(s-h)) = f(\chi(s) + \omega(s-h), z(s), z(s-h)) \left(\frac{-\mu(\chi(s) - \tilde{\eta}) + G(z(s))}{1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))} + 1 \right) \cdot \dot{\omega}(s-h) = f(\chi(s) + \omega(s-h), z(s), z(s-h)) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1} \cdot \dot{\omega}(s-h)$.

Hence (8), (9) can be rewritten for $s \in [0, h]$ (the first step) as

$$\begin{cases} \dot{z}(s) &= f(\chi(s) + \omega(s-h), z(s), z(s-h)) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1} \cdot \dot{\omega}(s-h), & s \in [0, h], \\ z(s) &= g(\omega(s)), & s \in [0, h], \\ \dot{\chi}(s) &= \{-\mu(\chi(s) - \tilde{\eta}) + G(z(s))\} \dot{\omega}(s-h) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1}, \\ \chi(0) &= \eta^0, \end{cases} \quad (15)$$

It easy to see that the last system gives solution $(z(s), \chi(s))$ for $s \in [0, h]$. Then the time transformation α is found by $\alpha(s) = \chi(s) + \omega(s-h)$. In general it reads as

$$\begin{cases} \dot{z}(s) &= f(\chi(s) + \alpha(s-h), z(s), z(s-h)) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1} \cdot \dot{\alpha}(s-h), & s \geq s^0, \\ z(s) &= g(\omega(s)), & s \in [s^0-h, s^0], \\ \dot{\chi}(s) &= \{-\mu(\chi(s) - \tilde{\eta}) + G(z(s))\} \dot{\alpha}(s-h) [1 + \mu(\chi(s) - \tilde{\eta}) - G(z(s))]^{-1}, & s \geq s^0, \\ \chi(s^0) &= \eta^0, \end{cases} \quad (16)$$

and can be solved step by step.

3. Connection between asymptotic properties of systems (1)-(4) and (8)-(9).

In this section we discuss how to determine if some qualitative properties of solutions of the initial state-dependent delay system (1)-(4) survive the time transformation i.e. still be valid for the corresponding solutions of constant delay system (8)-(9). We are also interested to connect the known properties of solutions of (8)-(9) with the ones of (1)-(4).

Let us start with the discussion of the property of the exponential stability. For the simplicity of presentation we assume that the zero function $y(t) \equiv 0$ is a solution of (1)-(4). Hence, by (11), $z(s) \equiv 0$ will be also a solution of (8)-(9).

Assume one has

$$\|z(s)\| \leq D^0 e^{-D^1(s-s^0)} \|z_{s^0}\|_{C([-h,0];R^m)}, s \geq s^0, D^0, D^1 > 0.$$

Hence, by (11), we get

$$\|y(t)\| \leq D^0 e^{-D^1(\alpha^{-1}(t)-s^0)} \|z_{s^0}\|_{C([-h,0];R^m)} = D^0 e^{-D^2(t-t^0)} \|z_{s^0}\|_{C([-h,0];R^m)} e^{D^2(t-t^0)-D^1(\alpha^{-1}(t)-s^0)}.$$

It is easy to see that if (and only if) $e^{D^2t-D^1\alpha^{-1}(t)}$ is bounded, then we have

$$\|y(t)\| \leq D^3 e^{-D^2(t-t^0)} \|z_{s^0}\|_{C([-h,0];R^m)}.$$

The above considerations show that the exponential stability of the zero solution $z(s) \equiv 0$ implies the exponential stability of the zero solution $y(t) \equiv 0$ provided there are positive constants D^1, D^2 such that $D^2t - D^1\alpha^{-1}(t) \leq 0$ for all $t \geq t^0$. Similar estimates give the inverse implication i.e. the exponential stability of the zero solution $y(t) \equiv 0$ implies the exponential stability of the zero solution $z(s) \equiv 0$ provided there are positive constants C^1, C^2 such that $C^2s - C^1\alpha(s) \leq 0$ for all $s \geq s^0$. Since by (11) $t = \alpha(s)$, we arrive to the following

Definition 1. We say that "s-time" and "t-time" are equivalent if there are constants $A^1 > 0, A^2 > 0, B^1 \in R, B^2 \in R$ such that $A^1t + B^1 \leq s \leq A^2t + B^2$.

Remark A. It is evident that in this case we also have $\frac{1}{A^2}s - \frac{B^2}{A^2} \leq t \leq \frac{1}{A^1}s - \frac{B^1}{A^1}$.

We saw that in the case of the equivalent "s-time" and "t-time" the property of the exponential stability survives under the time transformation. Another consequence of the equivalence is that $t \rightarrow +\infty$ if and only if $s \rightarrow +\infty$, which is clearly important for the study of long-time asymptotic behavior of solutions.

The last result suggests to study the notion of time-equivalence in detail. Let us try to find if in our case we have the equivalence. The rule (10) and (7) show that we need to analyse the function σ^{-1} . Using the property $\sigma(t) \geq t-h$ (bounded delay) and invertibility of σ , we get $\sigma^{-1}(\tau) \leq \tau+h$. Hence, by (10), one has $\alpha(s) = \sigma^{-1}(\alpha(s-h)) \leq \alpha(s-h) + h$. Particularly, $\alpha(h) \leq \alpha(0) + h, \alpha(2h) \leq \alpha(h) + h \leq \alpha(0) + 2h$, etc. Hence the property $\alpha(kh) \leq \alpha(0) + kh$ and the strict increase of α give the following estimate

$$\alpha(s) \leq (\alpha(0) + h) + s. \quad (17)$$

Since $\alpha(s) = t$, the estimate (17) means the lower bound in definition 1 and the upper bound in remark A (with $A^1 = 1, B^1 = -(\alpha(0) + h)$).

The supplemented bounds in definition 1 and remark A are less obvious. For the moment we do not claim that it is true in general case, but present an additional assumption which guarantees the bounds.

Let us assume that the value of delay is bounded from below by a positive constant, say $h_1 > 0$. More precisely, $\eta(t) \geq h_1 \in (0, \tilde{\eta}]$ for all $t \geq 0$. A sufficient condition for the last property is $|G(y)| \leq \mu|\tilde{\eta} - h_1|$ for all $y \in R^m$. Under the above condition we have $\sigma^{-1}(\tau) \geq \tau + h_1$. By (10), one has $\alpha(s) = \sigma^{-1}(\alpha(s-h)) \geq \alpha(s-h) + h_1$.

Particularly, $\alpha(h) \geq \alpha(0) + h_1, \alpha(2h) \leq \alpha(h) + h_1 \leq \alpha(0) + 2h_1$, etc. Hence the property $\alpha(kh) \leq \alpha(0) + kh_1$ and the strict increase of α give the following estimate

$$\alpha(s) \geq (\alpha(0) - h_1) + \frac{h_1}{h}s. \quad (18)$$

Combining (17) and (18) we conclude that "s-time" and "t-time" are equivalent (with $A^1 = 1, B^1 = -(\alpha(0) + h), A^2 = \frac{h}{h_1}, B^2 = -\frac{h}{h_1}(\alpha(0) - h_1)$ in definition 1 and remark A).

Having the equivalence proved we can use it to compare the asymptotic behavior of the corresponding dynamical systems (process), constructed by solutions of systems before and after the time transformations.

Let us consider an autonomous case of (1) i.e. the system (c.f. (1)-(4))

$$\dot{y}(t) = f^a(y(t), y(t - \eta(t))), \quad t > 0, \quad (19)$$

$$\dot{\eta}(t) = -\mu(\eta(t) - \tilde{\eta}) + G(y(t)), \quad t > 0, \quad (20)$$

with initial data

$$y(t) = g(t), \quad t \in [-h, 0], \quad (21)$$

$$\eta(0) = \eta^0. \quad (22)$$

We can restrict our study (using lemma 1) to the set of initial data

$$X_{fa} = \{(g, \eta^0) \mid \dot{g}(0) = f^a(g(0), g(-\eta^0))\} \subset C^1([-h, 0]; R^m) \times [0, h]. \quad (23)$$

The set X_{fa} is analogous to the *solution manifold* used in [15] (see also [9]). We notice that the reason for this restriction is to have C^1 smoothness of solution at zero i.e. $\dot{y}(0-) = \dot{g}(0-) = \dot{y}(0+) = f^a(g(0), g(-\eta^0))$. It is easy to see that X_{fa} is invariant.

We define the evolution operator $S^a(t) : X_{fa} \rightarrow X_{fa}$, associated to the system (19)-(22), by the formula $S^a(t)(g; \eta^0) = (y_t; \eta(t))$, where $(y; \eta)$ is the unique solution of (19)-(22). It is easy to see that under our assumptions the pair (S^a, X_{fa}) constitutes a dynamical system (in other words, the IVP (19)-(22) is well-posed in X_{fa}). For more definitions and details on dynamical systems see e.g. [8, 14, 4].

To discuss the properties of solutions to the *non-autonomous* system (8), (9), let us remind the following definition from [16, pp. 112-119]. Let E be a Banach space. Consider the two-parameter family of maps $\{U(t, \tau)\}$, $U(t, \tau) : E \rightarrow E$, parameters $\tau \in R, t \geq \tau$.

Definition 2. [16, p.113]. *A family of maps $\{U(t, \tau)\}$ is called a **process** on E if*

(i) $U(\tau, \tau) = I = \text{identity}$;

(ii) $U(t, s) \circ U(s, \tau) = U(t, \tau)$ for all $t \geq s \geq \tau \in R$.

Since (8), (9) has constant delay h we have no need to restrict our study to a solution manifold. We define

$$E = C([-h, 0]; R^m) \times [0, h] \times \\ \times \{\omega \in C^1[-h, 0] \mid \dot{\omega}(\cdot) > 0, \omega(-h) = \omega(0) - \eta^0; \dot{\omega}(0) [1 + \mu(\eta^0 - \tilde{\eta}) - G(\varphi(0))] = \dot{\omega}(-h)\}$$

and define $U(s, s^0)(\varphi; \eta^0; \omega) = (z_s; \chi(s); \alpha_s)$, where $(z; \chi; \alpha)$ is the unique solution of (8), (9) with initial data $(\varphi; \eta^0; \omega)$, here $z_{s^0} = \varphi$.

We notice that to come back to the system (19)-(22), using a particular solution of (8), (9), one restores function g as follows $g(t) = z_{s^0}(\alpha^{-1}(t))$ for $t \in [t^0 - h, t^0]$.

Let us continue to discuss which asymptotic properties of (1)-(4) survive the time transformation i.e. still be valid for the corresponding solutions of constant delay system (8)-(9).

One of the important properties of dynamical systems and processes are boundedness of solutions and their compactness (asymptotic compactness) see e.g. [8, 14, 4]. Below we always assume that $t \rightarrow +\infty$ if and only if $s \rightarrow +\infty$, which is true, for example, in the case of the equivalence of t -time and s -time. One can easily see, by (11), that $\|y(t)\| \leq C, t \geq t_1$ is equivalent to $\|z(s)\| \leq C, s \geq s_1$. Hence, the existence of a bounded absorbing set for z_s -coordinate is equivalent to the existence of a bounded absorbing set for y_t -coordinate, both in the space $C([-h, 0]; R^m)$. To go further, let us discuss the following additional assumptions on the time transformation α :

(A1) $\exists C^{1,\alpha} > 0, \forall s \geq s_1 \Rightarrow \dot{\alpha}(s) \leq C^{1,\alpha}$.

(A1') $\exists C^{2,\alpha} > 0, \forall t \geq t_1 \Rightarrow \dot{\alpha}^{-1}(t) \leq C^{2,\alpha}$.

(A2) α is uniformly continuous on $[s_1, +\infty)$.

(A2') α^{-1} is uniformly continuous on $[t_1, +\infty)$.

(A3) $\dot{\alpha}$ is uniformly continuous on $[s_1, +\infty)$.

(A3') $\dot{\alpha}^{-1}$ is uniformly continuous on $[t_1, +\infty)$.

Using (11), we have $\dot{z}(s) = \dot{y}(\alpha(s)) \dot{\alpha}(s)$. Hence, the existence of a bounded absorbing set for the y_t -coordinate in the space $C^1([-h, 0]; R^m)$ implies the existence of a bounded absorbing set for the z_s -coordinate, provided (A1) is satisfied. The inverse implication is valid provided (A1') is satisfied. For

the system (1)-(4) the existence of a bounded absorbing set means it is dissipative (for more details on this property see e.g. [8, 14, 4]). Let us now assume that for $t \geq t_1$ the y_t -coordinate belongs to a (pre-) compact set in the space $C([-h, 0]; R^m)$. By the Arzela-Ascoli theorem, the family $\{y_t\}_{t \geq t_1}$ is uniformly bounded and equicontinuous. Using (11), we have $|z(s^1) - z(s^2)| = |y(\alpha(s^1)) - y(\alpha(s^2))|$. This and the above discussion show that for $s \geq s_1$ the z_s -coordinate belongs to a (pre-) compact set in the space $C([-h, 0]; R^m)$, provided (A2) is satisfied. The inverse implication is valid provided (A2') is satisfied. The similar considerations in $C^1([-h, 0]; R^m)$ need the estimate $|\dot{z}(s^1) - \dot{z}(s^2)| = |\dot{y}(\alpha(s^1))\dot{\alpha}(s^1) - \dot{y}(\alpha(s^2))\dot{\alpha}(s^2)| \leq |\dot{y}(\alpha(s^1)) - \dot{y}(\alpha(s^2))||\dot{\alpha}(s^1)| + |\dot{y}(\alpha(s^2))||\dot{\alpha}(s^1) - \dot{\alpha}(s^2)|$. We see that if for $t \geq t_1$ the y_t -coordinate belongs to a (pre-) compact set in the space $C^1([-h, 0]; R^m)$, then for $s \geq s_1$ the z_s -coordinate belongs to a (pre-) compact set in the space $C^1([-h, 0]; R^m)$, provided (A1)-(A3) are satisfied. The inverse implication is valid provided (A1') - (A3') are satisfied. Particularly, we shown that assumptions (A1)-(A3) and (A1') - (A3') connect asymptotic properties of the dynamical system $(S^a(t), X_{f^a})$ and the process $U(s, \tau) : E \rightarrow E$.

Remarks. 1. Discussing assumptions (A1)-(A3), one could think that the family $\{\alpha_s\}_{s \geq s_1}$ may belong to a (pre-) compact set in the space $C[-h, 0]$ or even $C^1[-h, 0]$, but it is never true since $\alpha(s) = t$ is time, which is naturally unbounded.

2. One can see that (A1) implies (A2), but (A2) \nRightarrow (A1), similarly (A1') \Rightarrow (A2'), but (A2') \nRightarrow (A1').

3. We notice that (A1) gives $\alpha(s) \leq C^{1,\alpha}s + k^1$ and similarly (A1') $\Rightarrow \alpha^{-1}(t) \leq C^{2,\alpha}t + k^2$. Using definition 1, we see that (A1), (A1') imply the equivalence of "s-time" and "t-time".

Acknowledgments. The author thanks S. Maset for useful comments on an earlier version of the manuscript.

References

- [1] O. Arino, K.P. Hadeler, M.L. Hbid, Existence of periodic solutions for delay differential equations with state dependent delay, *Journal of Differential Equations*, Volume 144, Issue 2, 10 (1998), 263-301.
- [2] H. Brunner, S. Maset, Time transformations for delay differential equations, *Discrete and Continuous Dynamical Systems*, Vol. 25, Issue 3, (2009), 751-775.
- [3] H. Brunner, S. Maset, Time transformations for state-dependent delay differential equations, *Commun. Pure Appl. Anal.* 9 (2010), no. 1, 23-45.
- [4] I.D. Chueshov, "Introduction to the Theory of Infinite-Dimensional Dissipative Systems," Acta, Kharkov, 1999, (in Russian); English translation, 2002, <http://www.emis.de/monographs/Chueshov/>
- [5] O. Diekmann, S. van Gils, S. Verduyn Lunel, H-O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.
- [6] J.K. Hale, *Theory of Functional Differential Equations*, Springer, Berlin- Heidelberg- New York, 1977.
- [7] J.K. Hale and S.M. Verduyn Lunel, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1993.
- [8] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.
- [9] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Functional differential equations with state-dependent delays: Theory and applications. *In HANDBOOK OF DIFFERENTIAL EQUATIONS, Ordinary Differential Equations*, vol. 3, Canada, A., Drabek., P. and A. Fonda eds., pp. 435-545, Elsevier Science B. V., North Holland, 2006.
- [10] T. Krisztin, A local unstable manifold for differential equations with state-dependent delay, *Discrete Contin. Dyn. Syst.* 9 (2003), 993-1028.
- [11] J. Mallet-Paret, R. D. Nussbaum, P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, *Topol. Methods Nonlinear Anal.* 3(1), (1994), 101-162.

- [12] A.V. Rezounenko, Differential equations with discrete state-dependent delay: uniqueness and well-posedness in the space of continuous functions, *Nonlinear Analysis Series A: Theory, Methods and Applications*, Volume 70, Issue 11 (2009), 3978-3986.
- [13] A.V. Rezounenko, A condition on delay for differential equations with discrete state-dependent delay, *Journal of Mathematical Analysis and Applications*, 385 (2012,) 506-516.
- [14] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics и Physics*, Springer, Berlin-Heidelberg-New York, 1988.
- [15] H.-O. Walther, The solution manifold и C^1 -smoothness for differential equations with state-dependent delay, *Journal of Differential Equations*. -2003. no. 1. -46-65.
- [16] V. Chepyzhov, M.I. Vishik, Appendix: Non-athonomous dynamical systems and their attractors, *in* M.I. Vishik, *Asymptotic Behaviour of Solutions of Evolutionary Equations*, Cambridge University Press, 1992, 166 p.